Heat kernel estimates for symmetric jump processes with general mixed polynomial growths

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References

This talk is based on the following paper with Joohak Bae, Jaehoon Kang and Jaehun Lee (Seoul National University):

Heat kernel estimates for symmetric jump processes with mixed polynomial growths, arXiv:1804.06918 [math.PR]

Outline

For $u \in L^2(\mathbb{R}^d, dx)$, define a Dirichlet form $(\mathcal{E}, \mathcal{F})$ as

$$\mathcal{E}(u,v) := \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))(v(x) - v(y))J(x,y)dxdy$$

and

$$\mathcal{F} = \{f \in L^2(\mathbb{R}^d, dx) : \mathcal{E}(f,f) < \infty\}.$$

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$$\mathcal{F} = \{f \in L^2(\mathbb{R}^d, dx) : \mathcal{E}(f,f) < \infty\}.$$

Here J is a symmetric measurable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\}$.

Symmetric Hunt process

Under some mild assumptions on J, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is, in fact, a regular Dirichlet form,
i.e., $C_c(\mathbb{R}^d) \cap \mathcal{F}$ is dense in $\mathcal{F}$ with the norm $\mathcal{E}(f, f) + \int_{\mathbb{R}^d} |f|^2 dx$ and $C_c(\mathbb{R}^d) \cap \mathcal{F}$ is dense in $C_c(\mathbb{R}^d)$ with the uniform norm.

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Thus, by Fukushima (71) and Silverstein (74), there is a symmetric (conservative) Hunt process X in $\mathbb{R}^d$ associated with $(\mathcal{E}, \mathcal{F})$. Its L^2-infinitesimal generator $\mathcal{L}f(x)$ is

$$\mathcal{L}u(x) = \lim_{\varepsilon \downarrow 0} \int_{\{|y-x|>\varepsilon\}} (u(y) - u(x))J(x, y)dy.$$

Jumping Kernel for Symmetric Hunt process

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Let $\tau_D := \inf\{t > 0 : X_t \notin D\}$ be the first exit time from D by X.

For $A \subset \mathbb{R}^d \setminus \overline{D}$,

$$\mathbb{P}_x(X_{\tau_D} \in A) = \mathbb{E}_x \int_0^{\tau_D} \int_A J(X_t, z) dz dt$$

Heat Kernel estimates

Analytic point of view

$p(t, x, y)$ is called the heat kernel for $\mathcal{L}$ if

$$u(t, x) := \int_{\mathbb{R}^d} p(t, x, y) f(y) dy.$$

is the solution to

$$\mathcal{L}u = \partial_t u, \quad u(0, x) = f(x).$$

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Probabilistic point of view

$p(t, x, y)$ is the transition density function of X. i.e.,

$$\mathbb{P}_x(X_t \in A) = \int_A p(t, x, y) dy.$$

Thus obtaining sharp two-sided estimates for $p(t,x,y)$ is a fundamental problem in both analysis and probability theory.

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If $p(t,x,y)$ is a heat kernel for uniformly elliptic differential operator, then $p(t,x,y)$ enjoys

Gaussian estimates

$$c_1 t^{-d/2} e^{-c_2|x-y|^2/t} \le p(t,x,y) \le c_3 t^{-d/2} e^{-c_4|x-y|^2/t}.$$

History on Heat Kernel estimate for discontinuous Markov processes

While the heat kernel for diffusion process had been studied for more than a century, the heat kernel estimates for the discontinuous Markov process X (equivalently, for the non-local operator $\mathcal{L}$) have only been studied since around 2000.

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After pioneering works by Kolokoltsov (00), Bass & Levin (02), Chen & Kumagai (03), obtaining sharp two-sided estimates of heat kernels for various classes of discontinuous Markov processes becomes one of the most active topics in modern probability theory.

History on HKE for discontinuous Markov processes

- Chen & Kumagai (03, 08) : stalbe-like/mixed on metric measure space
- Bogdan & Jakubowski (07) : $\Delta^{\frac{\alpha}{2}} + B \cdot \nabla$
- Song & Vondraček (07) : $\Delta^{\frac{\alpha}{2}} + \Delta$
- Chen, K & Kumagai (08) : Finite range process
- Chen & Kumagai (10) : Diffusion with jump
- Sztonyk (10, 11) : Stable and tempered stable Lévy process
- Chen, K & Kumagai (11) : $J(x,y) \asymp e^{-|x-y|^{\beta}} |x-y|^{-d-\alpha}$
- Grigor'yan, Hu & Lau (14), Grigor'yan, Hu & Hu (17. 18+) : Analytic approach
- Bogdan, Grzywny &Ryznar (14) : Unimodal Lévy prcoess
- Sztonyk (13), Kateta & Sztonyk (13): derivatives Estimates
- Chen-Zhang (16–18+), Chen,Hu,Xie & Zhang (17) , K, Song & Vondraček (17), Kulczycki & Ryznar (17+), K & Lee (18+) : Non-symmetric Process
- Chen, Kumagai & Wang (16a,16b): Stability of Heat Kernel estimate
- Murugan & Saloff-Coste (15, 18+): Long range random walks.

Stable-like processes (Chen-Kumagai, 03)

For $0 < \alpha < 2$, define

$$J(x,y) = c(x,y)|x-y|^{-d-\alpha}$$

where $c(x,y)$ is a symmetric function on $\mathbb{R}^d \times \mathbb{R}^d$ that is bounded between two strictly positive constants, that is,

$$c^{-1} \le c(x,y) \le c \qquad \text{for -a.e. } x,y \in \mathbb{R}^d.$$

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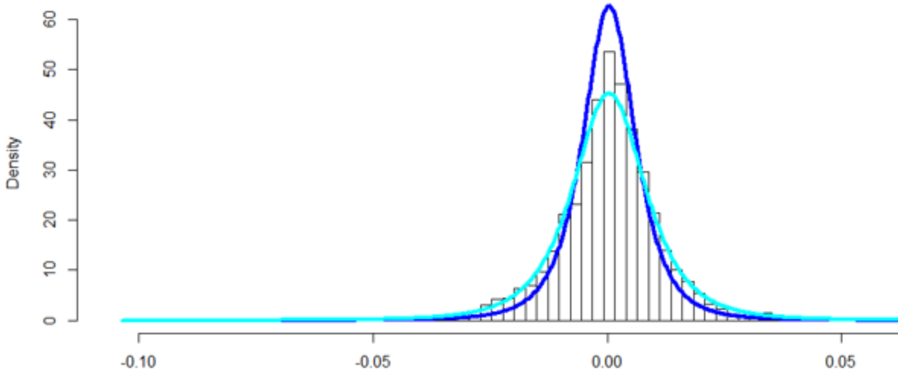
Chen-Kumagai, 03

$$p(t,x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}$$

for all $x, y \in \mathbb{R}^d$ and $t > 0$.

Heavy Tail



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Suppose

$$\frac{c^{-1}}{|x-y|^d\Phi(|x-y|)} \le J(x,y) \le \frac{c}{|x-y|^d\Phi(|x-y|)}\,, \quad x,y \in \mathbb{R}^d, \tag{1.1}$$

where Φ is a non-decreasing function on $[0,\infty)$ satisfying

$$c_1(R/r)^{\alpha_1} \le \Phi(R)/\Phi(r) \le c_2(R/r)^{\alpha_2}, \quad 0 < r < R < \infty \tag{1.2}$$

with $\alpha_1, \alpha_2 \in (0,2)$.

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with $\alpha_1, \alpha_2 \in (0,2)$.
Under the assumptions (1.1) and (1.2), $p(t,x,y)$ enjoys the following estimates.

Chen-Kumagai, 08

$$p(t,x,y) \asymp \Phi^{-1}(t)^{-d} \wedge tJ(x,y), \quad x,y \in \mathbb{R}^d, t>0.$$

Bogdan, Grzywny & Ryznar (14)

X is an isotropic unimodal pure-jump Lévy process in $\mathbb{R}^d$, i.e., the Lévy measure of X has a decreasing density $J(r)$ and the characteristic exponent of X is

$$\widehat{g}(|\xi|) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) J(|y|) dy.$$

Let $g(r) = \sup_{|x| \le r} \widehat{g}(|x|)$ and assume

$$c_1 (R/r)^{\alpha_1} \le g(R)/g(r) \le c_2 (R/r)^{\alpha_2}, \quad 0 < r < R < \infty$$

with $\alpha_1, \alpha_2 \in (0, 2)$.

Bogdan, Grzywny & Ryznar (14)

Let $\Phi(r) = 1/g(r^{-1})$. Then

$$p(t,x) \asymp \Phi^{-1}(t)^{-d} \wedge \frac{t}{|x|^d \Phi(|x|)} \quad \text{and} \quad J(|x|) \asymp \frac{1}{|x|^d \Phi(|x|)}$$

for all $x \in \mathbb{R}^d$ and $t > 0$.

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$$p(t,x,y) \asymp \Phi^{-1}(t)^{-d} \wedge \frac{t}{|x-y|^d \Phi(|x-y|)}.$$

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In both cases, we observe that the function Φ, which appears in the jumping kernel estimates, is *the scale function*, i.e., $|x-y| = \Phi(t)$ provides the borderline for $p(t,x,y)$ to have either near-diagonal estimates $\Phi^{-1}(t)^{-d}$ or off-diagonal estimates $\frac{t}{|x-y|^d \Phi(|x-y|)} \asymp tJ(x,y)$.

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Moreover, it is not difficult to show from the estimates that

$$c^{-1}\Phi(r) \le \mathbb{E}^z[\tau_{B(z,r)}] \le c\Phi(r) \quad \text{for all } \; z \in \mathbb{R}^d,\ r > 0,$$

where τ_A is the first exit time from A for the process X.

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Thus,

$$\frac{c^{-1}}{J(x,y)r^d} \le \mathbb{E}^z[\tau_{B(z,r)}] \le \frac{c}{J(x,y)r^d}, \text{ for all } r > 0 \text{ and } x, y, z \in \mathbb{R}^d \text{ with } |x-y| = r.$$

Goal

We investigate the estimates of transition densities of pure-jump symmetric Markov processes in $\mathbb{R}^d$, whose jumping kernels with general mixed polynomial growths, i.e.,

$$\frac{c^{-1}}{|x-y|^d\psi(|x-y|)} \le J(x,y) \le \frac{c}{|x-y|^d\psi(|x-y|)}\,, \quad x,y\in\mathbb{R}^d,$$

and

$$c_1(R/r)^{\alpha_1} \le \psi(R)/\psi(r) \le c_2(R/r)^{\alpha_2}, \quad 0<r<R<\infty$$

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As a corollary of the main result, we have the global sharp two-sided estimates when α_1 is greater than 1.

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In our settings, we only have

$$\mathbb{E}^z[\tau_{B(z,r)}] \lesssim\!\!\!\!\!\not\,\,\,\gtrsim \frac{c}{J(x,y)r^d}, \quad \text{for all } r>0 \text{ and } x,y,z \in \mathbb{R}^d \text{ with } |x-y|=r.$$

Subordinate Brownian motion

Let $W = (W_t)$ be a Brownian motion in $\mathbb{R}^d$ and
$S = (S_t)$ be an independent subordinator with Laplace exponent ϕ. i.e.,

$$\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}.$$

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Laplace exponent ϕ has a representation

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\,\mu(dt),$$

where $b \ge 0$ is called the drift, and $\mu(dt)$ is a measure satisfying $\int_0^\infty (1 \wedge t)\mu(dt) < \infty$, which is called the Lévy measure of ϕ.

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(Schilling, Song & Vondraček, *Bernstein functions*, 2nd ed., 2012)

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The characteristic exponent of subordinate Brownian motion is $\psi(\xi) = \phi(|\xi|^2)$.

Heat kernel estimates for Subordinate Brownian motion under weak scaling condition on ϕ

Suppose ϕ is a Bernstein function satisfying the following weak scaling condition: There exist $a_1, a_2 > 0$ and $\delta_1, \delta_2 \in (0, 2)$ satisfying

$$a_1 \lambda^{\delta_1/2} \phi(t) \le \phi(\lambda t) \le a_2 \lambda^{\delta_2/2} \phi(t) \,, \qquad \lambda \ge 1, t > 0 \,.$$

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Recall that the Lévy density $J(x)$ of the corresponding subordinate Brownian motion has the following estimates:

$$J(x) \asymp |x|^{-d} \phi(|x|^{-2}) = \frac{1}{|x|^d \Phi(|x|)},$$

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$$J(x) \asymp |x|^{-d}\phi(|x|^{-2}) = \frac{1}{|x|^d \Phi(|x|)},$$

and the transition density $p(t, x)$ of the corresponding subordinate Brownian motion has the following estimates:

$$p(t,x) \asymp \phi^{-1}(t^{-1})^{d/2} \wedge t|x|^{-d}\phi(|x|^{-2}) = \Phi^{-1}(t)^{-d} \wedge \frac{t}{|x|^d \Phi(|x|)}.$$

(Chen & Kumgagai (08), K., Song & Vondraček(12), Bogdan, Grzywny & Ryznar (14))

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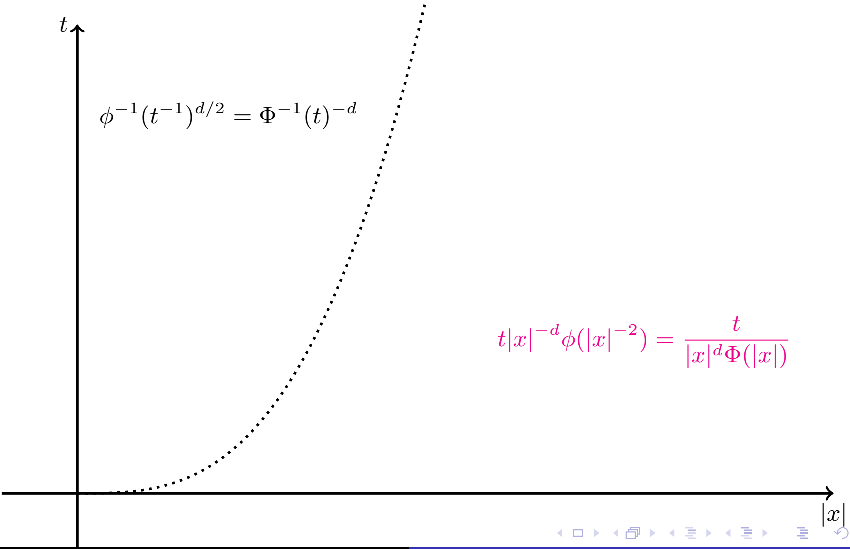

Brownian-like jump process (small jumps with high intensity): Mimica(16)

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We have

$$H(\lambda) := \phi(\lambda) - \lambda\phi'(\lambda) \asymp \begin{cases} \lambda^{1-\beta/2} & 0<\lambda<2 \\ \frac{\lambda}{(\log \lambda)^2} & \lambda \geq 2\,. \end{cases}$$

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$$\phi(\lambda) = \frac{\lambda}{\log(1+\lambda^{\beta/2})}, \quad \text{where } \beta \in (0,2).$$

Then ϕ varies regularly at ∞ with index 1 $(\lim_{t\to\infty} \phi(t\lambda)/\phi(t) = \lambda)$.

We have

$$H(\lambda) := \phi(\lambda) - \lambda\phi'(\lambda) \asymp \begin{cases} \lambda^{1-\beta/2} & 0 < \lambda < 2 \\ \frac{\lambda}{(\log \lambda)^2} & \lambda \geq 2\,. \end{cases}$$

Thus

$$\frac{\phi(\lambda)}{H(\lambda)} \asymp \log(\lambda), \quad \lambda \downarrow \infty.$$

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The function $H(\lambda) = \phi(\lambda) - \lambda\phi'(\lambda)$ appeared in Jain & Pruitt (87) to study asymptotic properties of tail probabilities of subordinators.

HKE for Subordinate Brownian motion, Mimica(16)

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HKE for Subordinate Brownian motion, Mimica(16)

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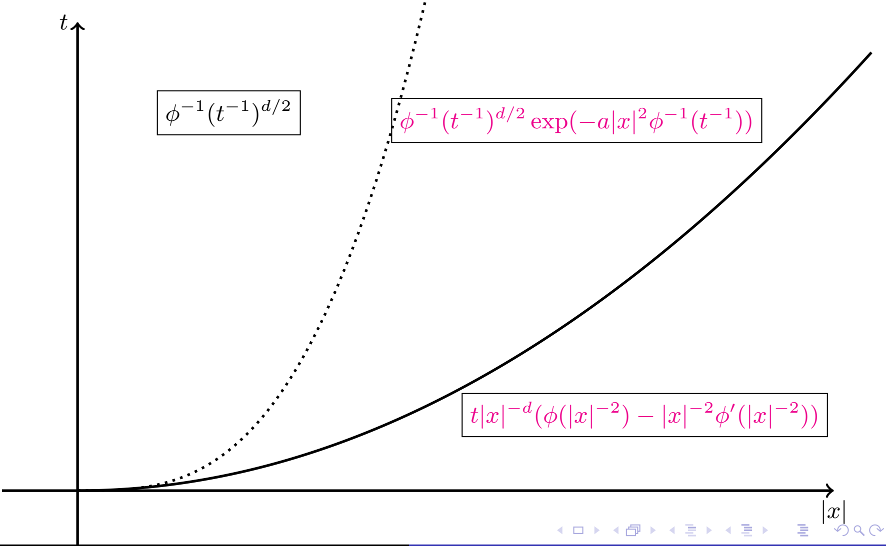
Suppose that ϕ has no drift and that

$$a_1\lambda^{\delta_1/2}H(t) \le H(\lambda t) \le a_2\lambda^{\delta_2/2}H(t)\,, \qquad \lambda \ge 1, t > 0 \quad (\text{ or } t > 1)\,,$$

with $\delta_1, \delta_2 \in (0, 4)$

Then the transition density $p(t, x)$ of X is comparable to

HKE for Subordinate Brownian motion, Mimica(16)



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If $0<t<1/2$ and $0<|x|<1/2$,

$$p(t,x) \asymp t^{-d/2}\left(\log\frac{1}{t}\right)^{-d/2} \wedge \left(\frac{t}{|x|^{d+2}(\log\frac{1}{|x|})^2} + t^{-d/2}\left(\log\frac{1}{t}\right)^{-d/2} \exp\left(-a\frac{|x|^2}{t}\log\frac{1}{t}\right)\right)$$

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For $t>1/2$ and $|x|>1/2$ we get stable-like estimates

$$p(t,x) \asymp t^{-d/(2-\beta)} \wedge \frac{t}{|x|^{d+2-\beta}}\,.$$

Weak scaling condition

Let $g : (0, \infty) \to (0, \infty)$, and $a \in (0, \infty]$, $\beta_1, \beta_2 > 0$, and $0 < C_L \le 1 \le C_U$.

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(1) For $a \in (0, \infty)$, we say that g satisfies $L_a(\beta_1, C_L)$ (resp. $L^a(\beta_1, C_L)$) if

$$\frac{g(R)}{g(r)} \ge C_L \left(\frac{R}{r}\right)^{\beta_1} \quad \text{for all} \quad r \le R < a \text{ (resp. } a \le r \le R).$$

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(3) When g satisfies $U_a(\beta, C_U)$ (resp. $L_a(\beta, C_U)$) with $a = \infty$, then we say that g satisfies the global weak upper scaling condition $U(\beta, C_U)$ (resp. the global weak lower scaling condition $L(\beta, C_L)$.)

Regular variation

For given function $f : (0, \infty) \to (0, \infty)$, we say that f varies regularly at 0 (resp. at ∞) with index $\delta_0 \in [0, \infty)$ if

$$\lim_{x \to 0} \frac{f(\lambda x)}{f(x)} = \lambda^{\delta_0} \quad (\text{resp. } \lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = \lambda^{\delta_0})$$

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Note that if f is non-increasing and is regularly varying at 0 (resp. at ∞) with index δ_0, then for any $a > 0$ and $0 < \underline{\delta} < \delta_0 < \bar{\delta}$, there is $C_U, C_L > 0$ such that f satisfies both $U_a(\bar{\delta}, C_U)$ and $L_a(\underline{\delta}, C_L)$ (resp. $U^a(\bar{\delta}, C_U)$ and $L^a(\underline{\delta}, C_L)$).

Throughout this talk, we will assume that $\beta_1, \beta_2 > 0$ and that $\psi : (0,\infty) \to (0,\infty)$ is a non-decreasing function satisfying $L(\beta_1, C_L)$, $U(\beta_2, C_U)$, and

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Let $J : \mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\} \to [0, \infty)$ be a symmetric function satisfying

$$\frac{C^{-1}}{|x-y|^d \psi(|x-y|)} \le J(x,y) \le \frac{C}{|x-y|^d \psi(|x-y|)}, \quad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus diag \tag{2.2}$$

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Note that (2.1) combined with (2.2) and $L(\beta_1, C_L)$ on ψ is a natural assumption to ensure that

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left(|x-y|^2 \wedge 1\right) J(x,y) dy \le c \left(\int_0^1 \frac{s ds}{\psi(s)} + \int_1^\infty \frac{ds}{s\psi(s)} \right) < \infty. \tag{2.3}$$

Scale function

We define our scale function as

$$\Phi(r) := \frac{r^2}{2\int_0^r \frac{s}{\psi(s)}\,ds}.$$

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In general, the function Φ is less than or equal to ψ. However, these functions may not be comparable unless $\beta_2 < 2$ where β_2 is the index in the weak upper scaling condition on ψ.

Remarks on Φ and ψ

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Suppose that ψ varies regularly at ∞ with index $\delta_0 \geq 2$. Then

$$\lim_{\lambda\to\infty} \frac{\Phi(\lambda)}{\psi(\lambda)} = 0,$$

and this implies that Φ varies regularly at ∞ with index 2.

Theorem 1.

There is a conservative Feller process $X = (X_t, \mathbb{P}^x, x \in \mathbb{R}^d, t \geq 0)$ associated with $(\mathcal{E}, \mathcal{F})$ that starts every point in $\mathbb{R}^d$. Moreover, X has a continuous transition density function $p(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$, with the following estimates: there exist $a_U, C > 0$ such that

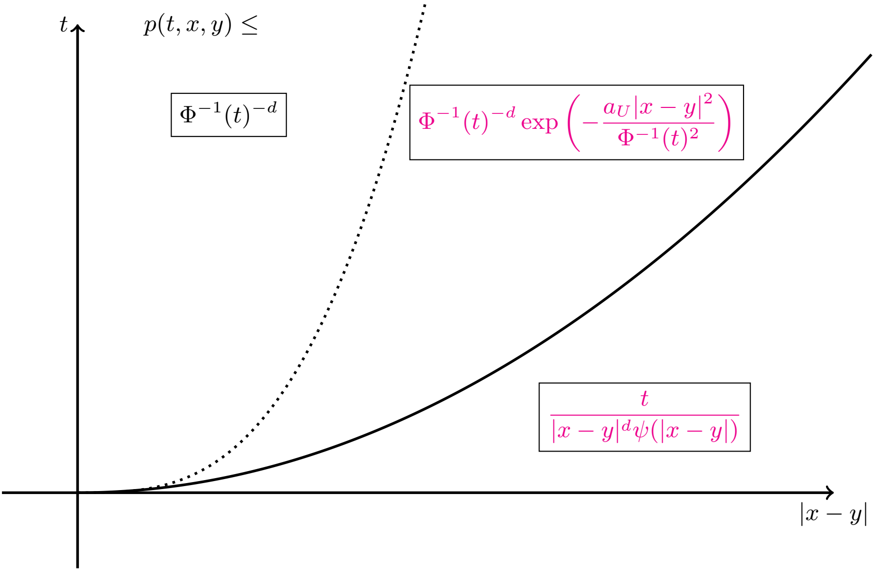
$$p(t,x,y) \leq C \left(\Phi^{-1}(t)^{-d} \wedge \left(\frac{t}{|x-y|^d \psi(|x-y|)} + \Phi^{-1}(t)^{-d} \exp\left(-\frac{a_U |x-y|^2}{\Phi^{-1}(t)^2} \right) \right) \right)$$

and

$$p(t,x,y) \geq C \left(\Phi^{-1}(t)^{-d} \wedge \frac{t}{|x-y|^d \psi(|x-y|)} \right).$$

In particular, if $\psi(r) \asymp \Phi(r)$ for all large $r > 1$, then for $t > 1$

$$p(t,x,y) \asymp \psi^{-1}(t)^{-d} \wedge \frac{t}{|x-y|^d \psi(|x-y|)}.$$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Suppose that $d > \beta_2 \wedge 2$ where β_2 is the index in the weak upper scaling condition $U(\beta_2, C_U)$ on ψ. Then there exists $c \geq 1$ such that for any $x, y \in \mathbb{R}^d$,

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If $d\geq 3$ and $\int_0^\infty \frac{s}{\psi(s)}\,ds<\infty$, then

$$c^{-1}|x-y|^{-d+2}\leq G(x,y)\leq c|x-y|^{-d+2},\quad |x-y|>1.$$

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$$p(t,x,y) \asymp \Phi^{-1}(t)^{-d} \wedge \left(\frac{t}{|x-y|^d \psi(|x-y|)} + \Phi^{-1}(t)^{-d} \exp\left(-\frac{c|x-y|}{\mathscr{K}^{-1}(t/|x-y|)} \right) \right). \tag{2.4}$$

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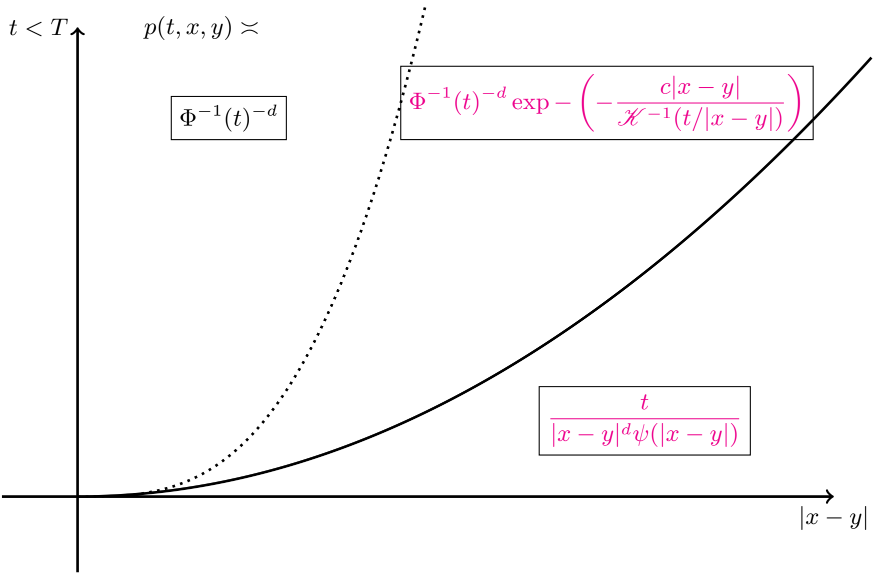
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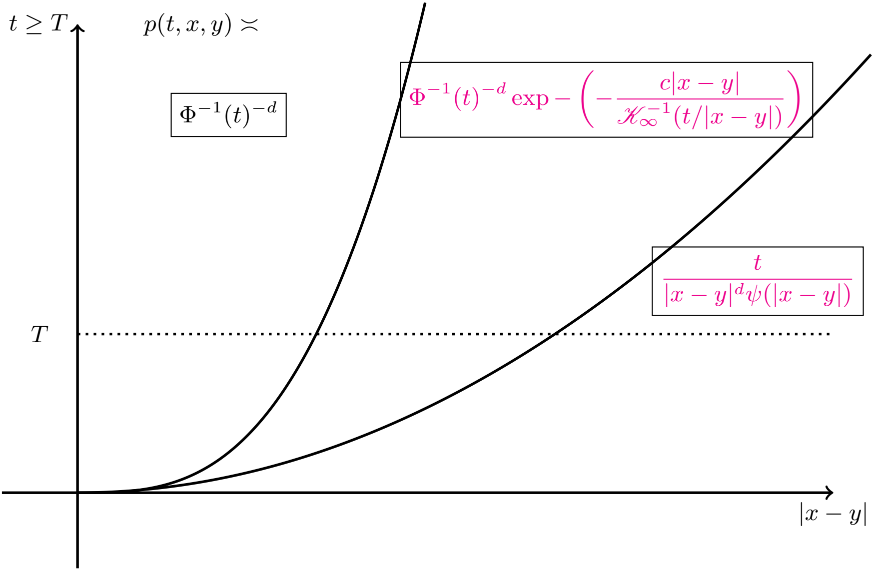
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Corollary: when $r \mapsto \Phi(r^{-1/2})^{-1}$ is a Bernstein function

Assume that Φ satisfies $L_a(\delta, \widetilde{C}_L)$ some $a > 0$ and $\delta > 1$, and $r \mapsto \Phi(r^{-1/2})^{-1}$ is a Bernstein function. Then, for any $T > 0$, there exist positive constants $c \geq 1$ and $a_U \leq a_L$ such that for all $(t, x, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d$,

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Moreover, if Φ satisfies $L(\delta, C_L)$ with $\delta > 1$, the estimates holds for all $t \in (0, \infty)$.

Finite second moment condition

These are all equivalent:

$$\sup_{x\in\mathbb{R}^d} \left(\text{or } \inf_{x\in\mathbb{R}^d}\right) \int_{\mathbb{R}^d} J(x,y)|x-y|^2 dy < \infty;$$

$$c^{-1}r^2 \le \Phi(r) \le cr^2, \quad r>1;$$

$$\int_0^\infty \frac{sds}{\psi(s)} < \infty;$$

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}_x[|X_t - x|^2] < \infty; \quad \text{for all } t>0;$$

and

$$\inf_{x\in\mathbb{R}^d} \mathbb{E}_x[|X_t - x|^2] < \infty \quad \text{for some } t>0.$$

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Gnedenko proved this result for the Lévy process in 1943 (see also Proposition 48.9 in Sato). The equivalence between the law of iterated logarithm and the finite second moment condition for the non-Lévy process has been a long standing open problem (see Shiozawa & Wang 2017 and references therein for previous works).

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Note that Euclidean space has the walk dimension 2; thus, the results in Chen, Kumagai & Wang (2016a+) does not cover our results.

By contrast, results under the upper bound of Jumping kernels ($J(x,y) \le c|x-y|^{-d}\Phi(|x-y|)^{-1}$) in Chen, Kumagai & Wang (2016a+) and Chen, Kumagai & Wang (2016b+), are applicable to our study.

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There exists a constant $C > 0$ such that for any $t > 0$ and $x, y \in \mathbb{R}^d$,

$$p(t,x,y) \leq C\left(\frac{1}{\Phi^{-1}(t)^d} \wedge \frac{t}{\Phi(|x-y|)|x-y|^d}\right)$$

We show the above rough upper bound, which enable us to use several main results in Chen, Kumagai & Wang (2016a+) and Chen, Kumagai & Wang (2016b+) to show parabolic Harnack inequality and the near-diagonal lower bound of $p(t,x,y)$.

Finite second moments

If X has the finite second moments, then $\mathscr{K}_\infty(r) \asymp r$ for $r > 1$.

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Thus, for $t > 1$,

$$c_1^{-1}\left(t^{-d/2} \wedge \left(tJ(x,y) + t^{-d/2}\exp\left(-c_2\frac{|x-y|^2}{t}\right)\right)\right)$$
$$\le p(t,x,y) \le c_1\left(t^{-d/2} \wedge \left(tJ(x,y) + t^{-d/2}\exp\left(-c_3\frac{|x-y|^2}{t}\right)\right)\right).$$

Suppose $\alpha > 1$ and

$$\psi(\lambda) \asymp \lambda^2 (\log \frac{1}{\lambda})^{\alpha}, \quad 0 < \lambda < 1/2,$$

that is,

$$J(x,y) \asymp \frac{1}{|x-y|^{d+2} (\log \frac{1}{|x-y|})^{\alpha}}, \quad |x-y| < 1/2.$$

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Then, for $t < 1/2$ and $|x-y| < 1/2$,

$$p(t,x,y) \asymp t^{-\frac{d}{2}} \left(\log \frac{1}{t}\right)^{\frac{d(\alpha-1)}{2}} \wedge \left(\frac{t}{|x-y|^{d+2} (\log \frac{1}{|x-y|})^\alpha} + t^{-\frac{d}{2}} \left(\log \frac{1}{t}\right)^{\frac{d(\alpha-1)}{2}} \exp\left(-a_2 \frac{|x-y|^2}{t} (\log \frac{1}{t})^{\alpha-1} \right) \right).$$

Suppose $\alpha > 1$ and

$$\psi(\lambda) \asymp \lambda^2 (\log \frac{1}{\lambda})(\log\log \frac{1}{\lambda})^{\alpha}, \quad 0 < \lambda < 1/16.$$

that is,

$$J(x,y) \asymp \frac{1}{|x-y|^{d+2}(\log \frac{1}{|x-y|})(\log\log \frac{1}{|x-y|})^{\alpha}}, \quad |x-y| < 1/16.$$

Then, for $t < 1/16$,

$$p(t,x,y) \asymp t^{-d/2}\left(\log\log \tfrac{1}{t}\right)^{\frac{d(\alpha-1)}{2}} \wedge \Bigg(\frac{t}{|x-y|^{d+2}(\log \frac{1}{|x-y|})(\log\log \frac{1}{|x-y|})^{\alpha}} + t^{-d/2}\left(\log\log \tfrac{1}{t}\right)^{\frac{d(\alpha-1)}{2}} \exp\left(-\tfrac{a_2|x-y|^2}{t}(\log\log \tfrac{1}{t})^{\alpha-1}\right)\Bigg).$$

Suppose $\beta \in \mathbb{R}$ and that $\psi(r) \asymp r^2(\log r)^\beta$, $r > 16$, that is,

$$J(x,y) \asymp \frac{1}{|x-y|^{d+2}(\log|x-y|)^\beta}, \quad |x-y| > 16.$$

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Then for $t \geq 16$,

(i) $\beta < 1$;

$$p(t,x,y) \asymp t^{-d/2}(\log t)^{-\frac{d(1-\beta)}{2}} \wedge \left(\frac{t}{|x-y|^{d+2}(\log(1+|x-y|))^{\beta}} + t^{-d/2}(\log t)^{-\frac{d(1-\beta)}{2}} \exp\left(-\frac{a_1|x-y|^2}{t(\log t)^{1-\beta}}\right)\right),$$

(ii) $\beta = 1$;

$$p(t,x,y) \asymp t^{-d/2}(\log\log t)^{-d/2} \wedge \left(\frac{t}{|x-y|^{d+2}\log(1+|x-y|)} + t^{-d/2}(\log\log t)^{-d/2} \exp\left(-\frac{a_2|x-y|^2}{t\log\log t}\right)\right),$$

(iii) $\beta > 1$;

$$p(t,x,y) \asymp t^{-d/2} \wedge \left(\frac{t}{|x-y|^{d+2}(\log(1+|x-y|))^{\beta}} + t^{-d/2} \exp\left(-a_3\frac{|x-y|^2}{t}\right)\right).$$

Thank you!

Main Lemma

Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function satisfying that $t \mapsto f(r,t)$ is non-increasing for all $r > 0$ and that $r \mapsto f(r,t)$ is non-decreasing for all $t > 0$. Fix $T \in (0, \infty]$. Suppose that the following hold:

(i) For each $b > 0$, $\sup_{t \le T} f(b\Phi^{-1}(t), t) < \infty$ (resp., $\sup_{t \ge T} f(b\Phi^{-1}(t), t) < \infty$);

(ii) there exist $\eta \in (0, \beta_1]$, $a_1 > 0$ and $c_1 > 0$ such that

$$\int_{B(x,r)^c} p(t,x,y)dy \le c_1 \left(\frac{\psi^{-1}(t)}{r} \right)^{\eta} + c_1 \exp\Big(-a_1 f(r,t) \Big)$$

for all $t \in (0,T)$ (resp. $t \in [T, \infty)$) and any $r > 0$, $x \in \mathbb{R}^d$.

Then, there exist constants $k, c_0 > 0$ such that

$$p(t,x,y) \le \frac{c_0\, t}{|x-y|^d \psi(|x-y|)} + c_0\, \Phi^{-1}(t)^{-d} \exp\Big(-a_1 k f(|x-y|/(16k), t) \Big)$$

for all $t \in (0,T)$ (resp. $t \in [T, \infty)$) and $x, y \in \mathbb{R}^d$.

There exist constants $a_1, C > 0$ and $N \in \mathbb{N}$ such that

$$p(t,x,y) \le \frac{C\,t}{|x-y|^d \psi(|x-y|)} + C\,\Phi^{-1}(t)^{-d} \exp\left(-\frac{a_1 |x-y|^{1/N}}{\Phi^{-1}(t)^{1/N}}\right),$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$.

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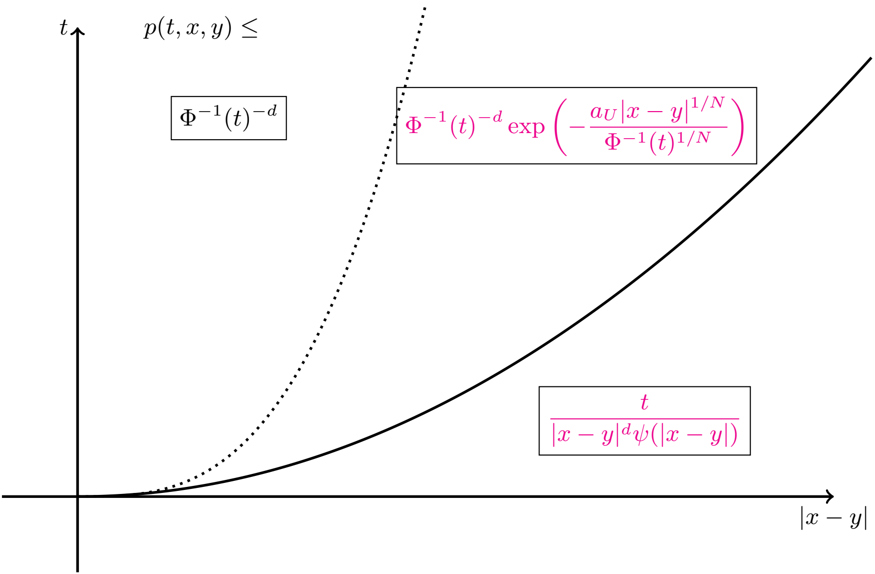
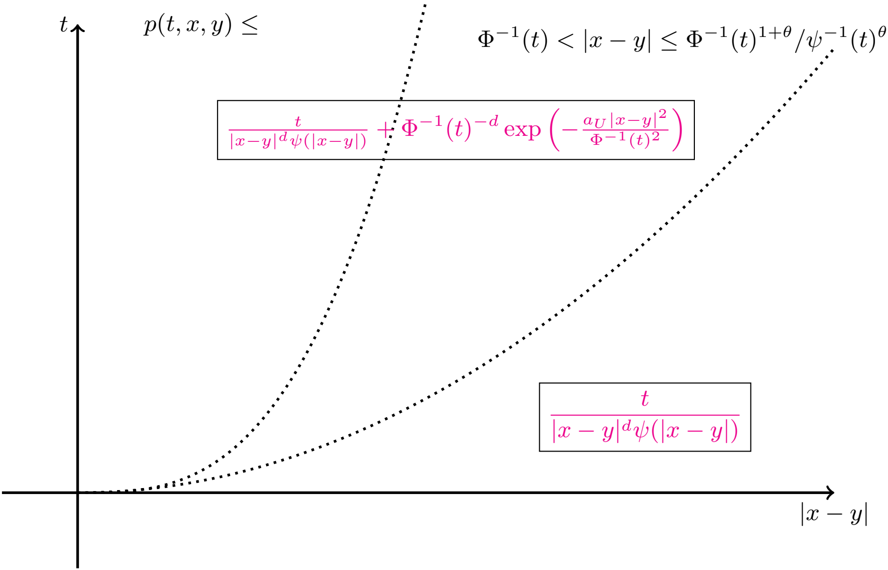
for all $t > 0$ and $x, y \in \mathbb{R}^d$.


t
$p(t,x,y) \leq$
$\Phi^{-1}(t)^{-d}$
$\Phi^{-1}(t)^{-d} \exp\left(-\frac{a_U |x-y|^{1/N}}{\Phi^{-1}(t)^{1/N}}\right)$
$\frac{t}{|x-y|^d \psi(|x-y|)}$
$|x-y|$


t
$p(t,x,y) \le$
$\Phi^{-1}(t) < |x-y| \le \Phi^{-1}(t)^{1+\theta}/\psi^{-1}(t)^{\theta}$
$\frac{t}{|x-y|^d \psi(|x-y|)} + \Phi^{-1}(t)^{-d} \exp\left(-\frac{a_U |x-y|^2}{\Phi^{-1}(t)^2}\right)$
$\frac{t}{|x-y|^d \psi(|x-y|)}$
$|x-y|$

Recall our notations:

$$\mathscr{K}(s) = \sup_{b \le s} \frac{\Phi(b)}{b}, \ \widetilde{\Phi}(t) = \Phi(1) t^2 \mathbf{1}_{\{0<t<1\}} + \Phi(t) \mathbf{1}_{\{t \ge 1\}}, \ \mathscr{K}_\infty(s) = \sup_{b \le s} \frac{\widetilde{\Phi}(b)}{b}.$$

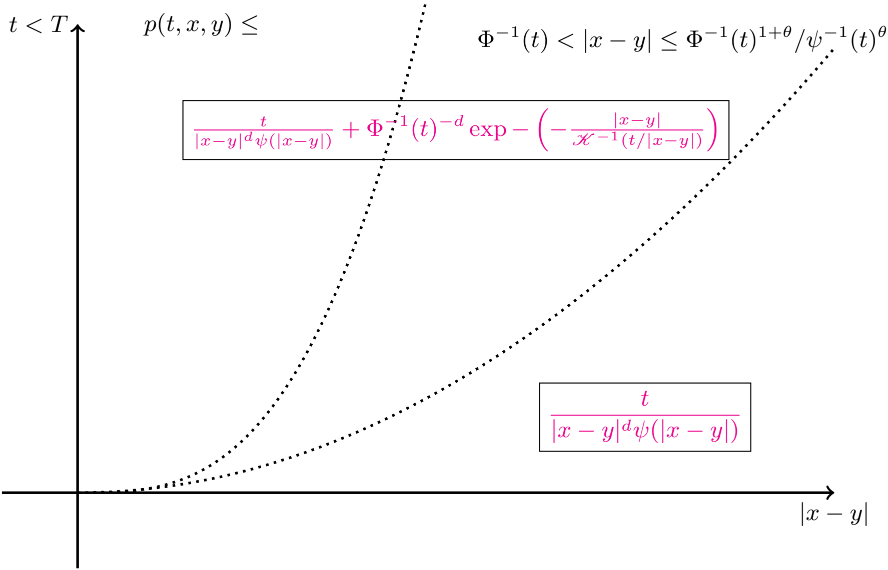
(1) Assume that Φ satisfies $L_a(\delta, \widetilde{C}_L)$ with $\delta > 1$. Then for any $T > 0$, there exist constants $a_U > 0$ and $c > 0$ such that for every $x, y \in \mathbb{R}^d$ and $t < T$,

$$p(t,x,y) \le \frac{c\,t}{|x-y|^d \psi(|x-y|)} + c\,\Phi^{-1}(t)^{-d} \exp\left(- \frac{a_U |x-y|}{\mathscr{K}^{-1}(t/|x-y|)} \right). \tag{4.1}$$

Moreover, if Φ satisfies $L(\delta, \widetilde{C}_L)$, then (4.1) holds for all $t < \infty$.

(2) Assume that Φ satisfies $L^1(\delta, \widetilde{C}_L)$ with $\delta > 1$. Then for any $T > 0$, there exist constants $a'_U > 0$ and $c' > 0$ such that for every $x, y \in \mathbb{R}^d$ and $t \ge T$,

$$p(t,x,y) \le \frac{c'\,t}{|x-y|^d \psi(|x-y|)} + c'\,\Phi^{-1}(t)^{-d} \exp\left(- \frac{a'_U |x-y|}{\mathscr{K}_\infty^{-1}(t/|x-y|)} \right).$$


$t < T$
$p(t,x,y) \leq$
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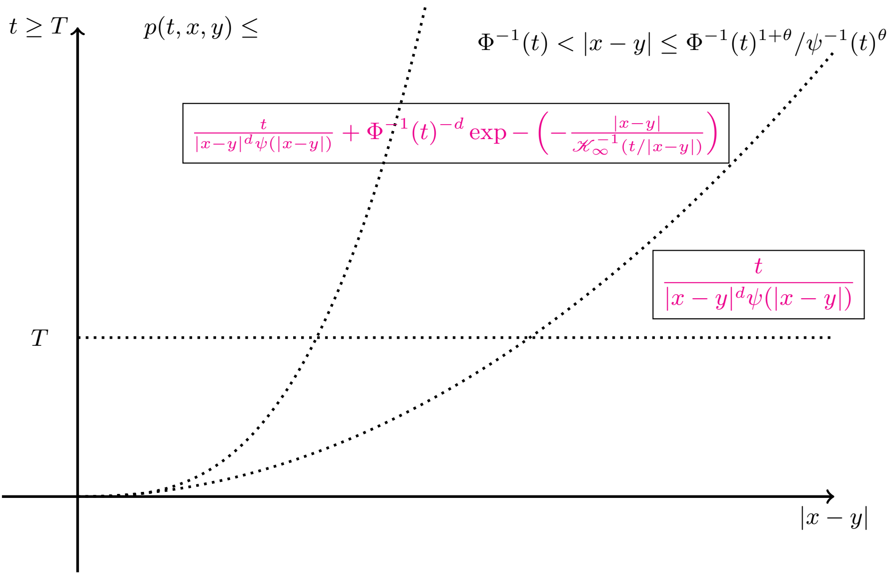
$\frac{t}{|x-y|^d\psi(|x-y|)}$
$|x-y|$


$t \geq T$
$p(t,x,y) \leq$
$\Phi^{-1}(t) < |x-y| \leq \Phi^{-1}(t)^{1+\theta}/\psi^{-1}(t)^{\theta}$
$\frac{t}{|x-y|^d \psi(|x-y|)} + \Phi^{-1}(t)^{-d} \exp - \left(- \frac{|x-y|}{\mathscr{K}_\infty^{-1}(t/|x-y|)} \right)$
$\frac{t}{|x-y|^d \psi(|x-y|)}$
T
$|x-y|$

Remarks on lower bound

Suppose Φ satisfies $L(\delta, \widetilde{C}_L)$ with $\delta > 1$ and for some $a > 0$. Then there exist $C > 0$ and $a_L > 0$ such that for any $t \le \Phi(|x-y|)$

$$p(t,x,y) \ge C\Phi^{-1}(t)^{-d} \exp\left(-a_L \frac{|x-y|}{\mathscr{K}^{-1}(t/|x-y|)}\right).$$

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$$p(t,x,y) \ge C\Phi^{-1}(t)^{-d} \exp\left(-a_L \frac{|x-y|}{\mathscr{K}^{-1}(t/|x-y|)}\right).$$

Proof. Recall $\mathscr{K}(s) = \sup_{b \le s} \frac{\Phi(b)}{b}$. Let $r = |x-y|$ and $k = \left\lceil 3r/\mathscr{K}^{-1}(\frac{t}{3r})\right\rceil$. By the definition of $\mathscr{K}$ and our choice of k,

$$\Phi\left(\frac{3r}{k}\right)\frac{k}{r} \le 3\mathscr{K}\left(\frac{3r}{k}\right) \le \frac{t}{r}.$$

Thus, we have $\frac{r}{k} \le \frac{1}{3}\Phi^{-1}(t/k)$.

Remarks on lower bound

Let $z_l = x + \frac{l}{k}(y-x)$, $l = 0, 1, \cdots, k-1$. For $\xi_l \in B(z_l, 3^{-1}\Phi^{-1}(\frac{t}{k}))$ and $\xi_{l-1} \in B(z_{l-1}, 3^{-1}\Phi^{-1}(\frac{t}{k}))$,
$|\xi_l - \xi_{l-1}| \le |\xi_l - z_l| + |z_l - z_{l-1}| + |z_{l-1} - \xi_{l-1}| \le \Phi^{-1}(t/k)$. Thus by the near diagonal lower bound, $p(\frac{t}{k}, \xi_{l-1}, \xi_l) \ge c_1 \Phi^{-1}(t/k)^{-d}$. Using the semigroup property, we get

$$
\begin{aligned}
&p(t,x,y) \ge \\
&\int_{B(z_{k-1}, 3^{-1}\Phi^{-1}(t/k))} \cdots \int_{B(z_1, 3^{-1}\Phi^{-1}(t/k))} p(\tfrac{t}{k}, x, \xi_1) \cdots p(\tfrac{t}{k}, \xi_{k-1}, y) d\xi_1 \cdots d\xi_{k-1} \\
&\ge c_1^k \Phi^{-1}(t/k)^{-dk} \prod_{l=1}^{k-1} \left| B(z_l, 3^{-1}\Phi^{-1}(t/k)) \right| \\
&= c_2 c_3^k \Phi^{-1}(t/k)^{-dk} \left(3^{-1}\Phi^{-1}(t/k)\right)^{d(k-1)} \\
&\ge c_2 (\frac{c_3}{3^d})^k \Phi^{-1}(t)^{-d} \ge c_2 \Phi^{-1}(t)^{-d} \exp\left(-(\log c)k\right) \\
&\ge c_2 \Phi^{-1}(t)^{-d} \exp\left(-c_4 \frac{r}{\mathscr{K}^{-1}(t/r)}\right).
\end{aligned}
$$